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New results on the Canham–Helfrich membrane model via the generalized Weierstrass representation

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Abstract

The Canham–Helfrich membrane model is discussed by making use of the generalized Weierstrass representation formulae for arbitrary surfaces immersed into the three-dimensional Euclidean space $\mathbb{R}^3$. Particular solutions of the shape equation are considered and correlators describing thermal fluctuations in the one-loop approximation are computed.

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1. Introduction

Surface models have a large domain of applicability in many fields of physics. In past years, their investigation has received an impressive input due to biophysics, where they naturally appear in connection with the study of biological membranes and vesicles (see, e.g., [1–10]). Precisely, the general interest has been attracted by lipid molecules, the major structural component of cell membranes, which in water spontaneously form bilayers in which hydrocarbon tails are shielded from contact with water cells. At length scales large compared with molecular dimension, the physics of a lipid bilayer membrane can be captured by projecting physical quantities on the bilayer midplane surface. Subsuming the membrane molecular architecture into a surface immersed into the three-dimensional Euclidean space $\mathbb{R}^3$ is what one traditionally means by the *mesoscale* approximation [1–10]. Within this approximation the free elastic energy of the membrane turns out to be the sum of terms which are proportional to scalar geometrical objects characterizing the associated midplane surface, such as area or curvatures.

Nowadays, the most effective geometric model for isolated membranes has proved to be the (generalized) Canham–Helfrich one, which can be derived from microscopic models and allows us to explain basic features and equilibrium shapes both for biological membranes and for liquid interfaces. As is well known, within this description one basically deals with the following free energy [1–8]

$$F_{\mathrm{CH}} = \int [\mathrm{d}\Sigma] \left[\tau + \frac{\kappa_c}{2}(H - 2c_0)^2 + \bar{\kappa} K \right] + \Delta P \int \mathrm{d}V \tag{1.1}$$

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where H is the mean curvature, K is the Gaussian curvature and $[\mathrm{d}\Sigma]$ is the Liouville area measure on the membrane surface Σ. Parameters τ, c_0, κ_c and $\bar{\kappa}$ are the surface tension, the spontaneous curvature, the bending rigidity and the Gaussian rigidity (or saddle-splay) coefficients respectively. Finally, in (1.1) a volume term $\Delta P \int \mathrm{d}V$ is also involved, ΔP denoting the pressure difference between the outside and the inside of the membrane. The free energy (1.1) is clearly reparametrization invariant. Physically, this follows from the mesoscale approximation in that it means that the membrane has no internal structure and is just made of a two-dimensional fluid with constant density. In order to stress the presence of a leading term with nonvanishing bending rigidity, membranes described through (1.1) shall be referred to afterwards as *rigid* membranes.

In general, the extended Canham–Helfrich model (1.1) can be read as an ordinary two-dimensional field theory (e.g., [5, 6]). Its study therefore covers the analysis both of geometrical aspects, which are encoded in the associated Euler–Lagrange equations, and of thermal equilibrium properties, which are encoded in the partition function written as the sum

$$Z = \int [\mathrm{D}\hat{\Sigma}] \exp\left[-\frac{F_{\mathrm{CH}}}{K_B T}\right] \tag{1.2}$$

over all membrane configurations $\hat{\Sigma}$. As regards the geometric architectures shown in the self-assembling of physical systems described by models of type (1.1), most of the current understanding relies on results which are known from the classical differential geometry. Basically, the surfaces usually discussed in the context of membrane mesoscale physics belong to the following classes: (i) minimal surfaces (see, e.g., [11]), (ii) Willmore surfaces (see, e.g., [12]) and (iii) constant mean curvature surfaces (see, e.g., [13]). A variety of basic minimal, Willmore and constant mean curvature geometries can be found in standard literature, and in the context of physical systems described by membrane models they have been discussed, for instance, in [6, 7, 14, 15] (see also references therein) respectively. Concerning thermal effects, there is a disagreement on some physical predictions. For many years, membrane deformations under thermal fluctuations have been widely described in terms of normal displacements from a reference plane parametrized by Cartesian coordinates (the so-called Monge gauge). In this framework rigid membranes have been predicted to be softened by thermal fluctuations. Nevertheless, it has been recently argued that a local mean curvature measure should be considered as the right statistical measure for the partition function [16]. In this case, in contrast, thermal fluctuations would stiffen the membrane. Also, due to the lack of experimental information in this respect, the adoption of the proper statistical measure for the partition function is still an open problem.

In this paper, we argue on some aspects of systems described by the free energy (1.1) by resorting to recent progress achieved in the differential geometry of surfaces. With this scope, we adopt a basic point of view and formulate some principal statements which provide quite a new setting to handle the subject. Following Konopelchenko [17], we make use of the so-called generalized Weierstrass representation (GWR) for surfaces in the three-dimensional Euclidean space $\mathbb{R}^3$. This geometrical setting may provide us with an effective picture and may be helpful in the attempt to have some further insight into the matter. We can proceed, in fact, without referring to the coordinates of the three-dimensional Euclidean ambient space in which membranes live or to their possible coupling to an auxiliary two-dimensional metric. The compact form for extrinsic curvature energy terms, as well as the linear form of the compatibility equations characterizing the GWR, makes it possible to describe statistical fluctuations of arbitrary membranes in a comfortable way. Yet, fluctuations of both intrinsic (the internal strain) and extrinsic (the location in the target space) nature can simultaneously be considered. (Recall that the strategy pursued in [16] was to first evaluate the leading term

for the mean-square strength of the bending rigidity under the hypothesis of a fixed area and next to take into account the effect of tilt of the membrane by reducing the area of molecules while keeping $\langle H^2 \rangle$ fixed.) This shall also clarify how basic geometric quantities enter the statistical fluctuations.

The paper has four other sections and an appendix. The outline is as follows. Section 2 is a rather brief introduction to the Weierstrass representation for surfaces in $\mathbb{R}^3$, with the equations and formulae required to provide the necessary background and to make the paper self-contained. In section 3 we focus on non-compact membrane geometries which are allowed within the Canham–Helfrich model and which could reveal themselves rather interesting. The statistical mechanics of the model is developed and discussed in section 4. Although it follows directly from the Canham–Helfrich prescriptions, it is primarily discussed as a somehow new model. We deal with the effective one-loop Canham–Helfrich free energy which follows after integrating the fluctuations of Weierstrass fields around classical backgrounds and derive the corresponding correlators in the one-loop approximation. After section 5, devoted to conclusions, we shall make in the appendix some comments recalling some known facts on constant mean curvature geometries.

2. The generalized Weierstrass representation for surfaces in $\mathbb{R}^3$

The study of surfaces is a central topic in mathematics. Their characterization, classification, description as well as the understanding of both their local and global properties represent, indeed, the main subject of the classical differential geometry and, perhaps, the most studied subject in the whole of mathematics in the last two centuries. In this section we are basically concerned with the problem of realizing conformal immersions of surfaces into $\mathbb{R}^3$. As anticipated, the basic tool of our approach to the membrane model (1.1) is the so-called generalized Weierstrass representation (GWR) for surfaces in $\mathbb{R}^3$ [18]. Since it has been introduced recently in connection with the theory of integrable systems and may be somewhat unfamiliar, the aim of this section is to give a short self-consistent introduction to the subject. Further details can be found in [18–22].

The basic idea underlying the GWR is to characterize a surface by focusing on its tangent map, or better on its Gauss map. For a surface in the three-dimensional Euclidean space $\mathbb{R}^3$, the Gauss map is defined as the mapping of tangent planes to the surface into the Grassmannian manifold $G_{2,3}$, which in turn is realized as a quadratic in CP^1 (see [29]). The key point is that, as shown by Kenmotsu in [30], an arbitrary surface in $\mathbb{R}^3$ with nonvanishing prescribed mean curvature is essentially uniquely determined by its Gauss map. However, since not every field in $G_{2,3}$ forms a tangent plane to the surface then a proper integrability condition must be introduced. The Kenmotsu theorem in [30] provides us with both the condition for the existence of a surface with prescribed mean curvature in the form of a nonlinear differential constraint involving the Gauss map and a simple integral representation for realizing the immersion into $\mathbb{R}^3$.[1]

In [18] Konopelchenko found a way to define the Gauss map of surfaces in $\mathbb{R}^3$ which, being equivalent, turns out to be simpler with respect to the Kenmotsu prescriptions. Konopelchenko's integrability condition is given by the linear system

$$\partial\psi = p\varphi \qquad \bar{\partial}\varphi = -p\psi \tag{2.1}$$

[1] A minimal surface into $\mathbb{R}^3$ cannot be uniquely determined by the Gauss map: the same Gauss map arises as the Gauss map of infinitely many different minimal surfaces. This aspect helps to clarify the importance of Kenmotsu's result.

where $\partial = \frac{\partial}{\partial z}$, $z \in G$ is a complex variable, $G \in \mathbb{C}$ is a simply connected domain in the complex plane[2], $\psi(z, \bar{z})$, $\varphi(z, \bar{z})$ are complex-valued functions while $p(z, \bar{z})$ is a real-valued function. Once the potential p and solutions ψ, φ of the system (2.1) are given, then one may define the mapping $X^\mu : G \to \mathbb{R}^3$ according to

$$X_1 + \mathrm{i}X_2 = 2\mathrm{i}\int_\Gamma [\bar{\psi}^2\,\mathrm{d}z' - \bar{\varphi}^2\,\mathrm{d}\bar{z}'] \qquad X_3 = -2\int_\Gamma [\bar{\psi}\varphi\,\mathrm{d}z' + \psi\bar{\varphi}\,\mathrm{d}\bar{z}'] \tag{2.2}$$

where Γ is a contour in the domain G ($z \in G$). By virtue of (2.1), the integrands in (2.2) are closed forms and integrals (2.2) are path independent. If one now treats the above $X^\mu(z, \bar{z})$ as surface coordinates, then formulae (2.1)–(2.2) give the conformal immersions into $\mathbb{R}^3$ of surfaces, $z, \bar{z} \in G$ being the conformal local coordinates. The pair ψ, φ gives the Gauss map while assigning the potential p amounts to assigning the mean curvature (see formulae (2.4)–(2.5) below). Every regular surface immersed into $\mathbb{R}^3$ admits a representation of the form (2.1)–(2.2), referred to as the generalized Weierstrass representation [18–28]. It is straightforward to note, indeed, that when the potential of the representation $p(z, \bar{z})$ vanishes one gets from (2.1) a holomorphic and an anti-holomorphic function, thus obtaining the classical Weierstrass representation of minimal surfaces into $\mathbb{R}^3$ (see, e.g., [11]).

By adopting the GWR formulae (2.1)–(2.2), geometric characteristics of surfaces take a rather compact form. For instance, the induced metric on Σ and the Gauss curvature read

$$\mathrm{d}s^2 = 4(|\psi|^2 + |\varphi|^2)^2\,\mathrm{d}z\,\mathrm{d}\bar{z} = 4u^2\,\mathrm{d}z\,\mathrm{d}\bar{z} \qquad K = -\frac{1}{u^2}\partial\bar{\partial}\,\log(u) \tag{2.3}$$

respectively while the mean curvature vector $\vec{H} = g^{z\bar{z}}\partial\bar{\partial}\vec{X}$ is

$$\vec{H} = \frac{p}{u^2}\{\mathrm{i}(\bar{\psi}\bar{\varphi} - \varphi\psi), (\bar{\psi}\bar{\varphi} + \varphi\psi), (|\psi|^2 - |\varphi|^2)\}. \tag{2.4}$$

Hence the squared mean curvature and the Willmore functional, defined as $W \stackrel{\mathrm{def}}{=} \int \vec{H}^2[\mathrm{d}\Sigma]$ (see [12]), are simply ($x = \mathrm{Re}\,z$, $y = \mathrm{Im}\,z$)

$$\vec{H}^2 = \frac{p^2}{u^2} \qquad W = 4\int p^2\mathrm{d}x\,\mathrm{d}y. \tag{2.5}$$

Since it gives the possibility of constructing a number of new examples of surfaces having interesting properties, the GWR representation (2.1), (2.2) might reveal itself as an effective tool when dealing with concrete problems and applications. In fact, by virtue of the linearity of the generating linear problem (2.1) and the simple structure of formulae (2.2) for coordinates, classes of surfaces of interest in mathematics, in physics and in computer-aided geometrical design can be easily characterized. For our purpose here, it turns out to be useful to make a brief digression before concluding the section by pointing out that, in the attempt to exploit its linearity when $p \neq 0$, we can consider the solutions to the system (2.1) of the form

$$\psi = \sum_\alpha \psi_\alpha = \sum_\alpha \exp[F^\alpha(z, \bar{z})] \qquad \varphi = \sum_\alpha \varphi_\alpha = \sum_\alpha p^{-1}[F^\alpha(z, \bar{z})]_z\psi_\alpha \tag{2.6}$$

where the complex functions F^α satisfy the differential equation

$$\left(pF^\alpha_{z\bar{z}} - F^\alpha_z\,p_{\bar{z}}\right) = -p\left(p^2 + F^\alpha_z F^\alpha_{\bar{z}}\right). \tag{2.7}$$

Two main classes of non-trivial solutions naturally arise. They correspond to (a) the separable Weierstrass potential case, $p(z, \bar{z}) = \epsilon A(z)A(\bar{z})$, for which $F^\alpha = F^\alpha_1(z) + F^\alpha_2(\bar{z})$ and (b) the one-dimensional reduction $p = p(\rho)$ (where $\rho = \mathrm{Re}\,z$ or $\rho = \mathrm{Im}\,z$), which implies

[2] As for the old Weierstrass formulae for minimal surfaces in $\mathbb{R}^3$ and for the Kenmotsu representation of arbitrary surfaces $\mathbb{R}^3$, the simple connectivity is required since one has integrals involving the Gauss map and the mean curvature; the integrals are taken over paths and there may be periods over curves not homologous to zero.

$F^\alpha = F^\alpha(\rho)$. In both cases, one can easily consider two classes of surfaces such that their induced metric takes a constant value. They correspond both to the single- and double-wave forms for the Weierstrass functions ψ and φ in (2.6). We will make use of this naive observation in the next section.

3. Classical configurations of rigid membranes

The characterization of configurations exhibited by rigid membranes is a relevant topic. Several systems have been observed experimentally to self-assembly into a variety of geometric structures which should be described in terms of rigid surfaces, at least in some limits. Although many aspects of these systems have been addressed in the past, only partial attention has been devoted to aspects inherent in the exact analytical description of configurations. This is for a reason of technical nature. Membrane configurations are described by a strongly nonlinear system, consisting of the Gauss–Codazzi–Mainardi equations (see, e.g., [12]) and the so-called shape equation (see, e.g., [8]). The latter expresses the Euler–Lagrange equation for extremals of the free energy (1.1) in terms of basic geometrical quantities on the membrane surface and reads [1–8]

$$2\kappa_c \Box H + \kappa_c(2H + c_0)[2H^2 - c_0 H - 2K] - 2\tau H + \Delta P = 0 \tag{3.1}$$

where $\Box = (\sqrt{\det g})^{-1}\partial_a \sqrt{\det g}\, g^{ab}\partial_b$ is the Laplace–Beltrami operator, g_{ab} being the induced metric on the membrane surface. Equation (3.1) determines the equilibria for closed or theoretically infinite Canham–Helfrich membranes. In these cases boundary conditions can be neglected and the saddle-splay $\bar{\kappa}$-term drops out of the shape equation. (For a recent discussion about boundary conditions, see [31].)

The system of (3.1) and Gauss–Codazzi–Mainardi equations is quite difficult to handle. Alternatively, one may rewrite the equation for a rigid surface in terms of the coordinate X^μ describing the surface in the target space $\mathbb{R}^3$. In this way, however, one gets a highly nonlinear equation involving fourth-order derivatives of membrane coordinates in the three-dimensional Euclidean space $\mathbb{R}^3$. As a consequence, the geometrical analysis of systems described by (1.1) has been mainly concerned with a few very special and well-known geometries, mostly related to one-dimensional limits of the shape equation (3.1) (see [8] and references therein). Since it is accepted that an exhaustive analytical treatment cannot be achieved, in past years efforts have been conceived in the direction of obtaining information by means of numerical methods, computer simulations and specific software (such as Brakke's surface evolver [32]). This approach provides a straightforward way of visualizing effects due to changes in physical parameters (see, for instance, [33, 34]). Nevertheless, this way proceeding may require a suitable knowledge of analytical aspects concerning reasonable membrane configurations.

As already stated, concerning the study of membrane geometries most of the attention has been paid to minimal surfaces, CMC surfaces and Willmore surfaces. In this section we would like to point out that, while attempting to go further in looking for new basic analytical formulae for membrane geometries, configurations of the *developable* type are of interest too. Developable surfaces are such that their Gaussian curvature is identically zero, $K = 0$, and play a distinguished role in the differential geometry of surfaces. Geometrically, the tangent plane to a developable surface is constant along a fixed ruling. The main motivation for considering these geometries comes from the experimental observation that lipids in water can self-assemble into large stable sheets of few tens of Å thick to form a variety of lamellar phases as a function of temperature and humidity. Four bilayer thermodynamic phases can be distinguished: a conventional liquid-crystal phase (L_α), a crystalline phase (L_c), a gel phase

($L_{\beta'}$) and a ripple phase ($P_{\beta'}$). The latter, in particular, has been a cause of considerable interest in recent years (see, e.g., [35, 36] and references therein). In the ripple phase, the lipid bilayer is distorted by a periodic ripple characterized by a two-dimensional monoclinic lattice. The resulting corrugated system can be either asymmetric or symmetric. The thermodynamical and statistical mechanics of lipid phase transitions is a rather intriguing subject and is well recognized as a challenging phenomenon for physical theories due to the present lack of knowledge of structural features of lipid phases and of the many parameters that may be involved. A simple purely geometrical idealized model fails to be useful since it would not allow us to straightforwardly take into account some mechanisms at a more fundamental level on the lipid bilayer, such as a sliding of the lipid molecules leading to an increased exposure of the lipid headgroups to the hydrophilic medium (see, e.g., [37]). However, if we are not strictly interested in phase transitions and in asking questions concerning what causes these changes to occur but, rather, are interested only in lamellar geometry, we can continue to apply the model (1.1). Geometrically, in the simplest case the ripple phase $P_{\alpha'}$ is nothing but a surface with one principal curvature vanishing and the other periodic; in other words, a surface with vanishing Gaussian curvature and periodic mean curvature. The existence of these surfaces in the context of the Canham–Helfrich model (1.1) is obviously not surprising. Surfaces with periodicity properties are indeed implied by the shape equation (3.1). On the other hand, on general grounds one can expect that within the class of periodic surface solutions to the shape equation, developable ones turn out to be energetically favoured for the formation of stable configurations. In the next part of this section we will thus discuss this case, characterizing some open-like solutions to the shape equation. In doing so, we make use of the Weierstrass representation (2.1), (2.2) showing how, in principle, one might hope to take full advantage by exploiting the linearity of the compatibility conditions (2.1) in the way outlined at the end of section 2.

3.1. Examples of periodic developable rigid membranes

In terms of the generalized Weierstrass representation, equations (2.1), (2.2), the vanishing of Gaussian curvature constrains the Weierstrass fields ψ, φ to satisfy the condition

$$|\psi|^2 + |\varphi|^2 = A(z)\overline{A(z)} \tag{3.2}$$

where $A(z)$ is an arbitrary holomorphic function. The general case of a developable Canham–Helfrich rigid membrane is thus described through the system made of equations (2.1), (3.2) and

$$2\kappa_c \partial\bar{\partial} H + \{k_c(2H + c_0)[2H^2 - c_0 H] - 2\tau H + \Delta P\}|A(z)|^4 = 0. \tag{3.3}$$

Below, we shall focus on the one-dimensional reduction for the potential of the Weierstrass representation, $p = p(\mathrm{Re}\, z) = p(x)$, for simplicity restricting to the case $A(z) =$ constant (which amounts to setting one of the X^μ to be linear in $y = \mathrm{Im}\, z$). The Weierstrass potential $p(x)$ should therefore satisfy the equation ($\partial = (\partial_x - \mathrm{i}\partial_y)/2$)

$$\partial_x^2 p + 8p^3 - 2c^4\left(c_0^2 + 2\tau\kappa_c^{-1}\right)p + 2\kappa_c^{-1}c^6\Delta P = 0 \tag{3.4}$$

c being a real constant. Hence, the Weierstrass potential turns out to be defined in terms of standard elliptic functions. It is easy to see that this one-dimensional limit can be related to Weierstrass functions of both types

$$\psi = \mathrm{i}\varphi \qquad \varphi = \frac{c}{\sqrt{2}}\exp\left[2\mathrm{i}\int p(x)\,\mathrm{d}x\right] \tag{3.5}$$

and

$$\psi = c_1 \sin\left[2\int^x p(x')\,\mathrm{d}x'\right] \qquad \varphi = c_1 \cos\left[2\int^x p(x')\,\mathrm{d}x'\right] \tag{3.6}$$

where c and c_1 are real constants. The above Weierstrass fields are related to surfaces of the type

$$\begin{aligned} X_1 &= X_1^0 - 2c^2 \int \mathrm{d}x \sin\left\{2\int^x p(x')\,\mathrm{d}x'\right\} \\ X_2 &= X_2^0 - 2c^2 \int \mathrm{d}x \cos\left\{2\int^x p(x')\,\mathrm{d}x'\right\} \\ X_3 &= X_3^0 - 2c^2 y \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} X_1 &= X_1^0 - c^2 y \\ X_2 &= X_2^0 + 2c^2 \int \mathrm{d}x \left\{\left[\cos 2\int^x p(x')\,\mathrm{d}x'\right]^2 - \frac{1}{2}\right\} \\ X_3 &= X_3^0 - 2c^2 \int \mathrm{d}x \left\{\sin 4\int^x p(x')\,\mathrm{d}x'\right\} \end{aligned} \tag{3.8}$$

respectively. Formulae (3.7), (3.8) thus provide us with lamellar periodic geometries which take place in the membrane Canham–Helfrich model (1.1). They are characterized by a periodic one-dimensional height modulation which reminds us of those showing up in the ripple phase of some phospholipids under high hydration [35, 36]. Formulae (3.7), (3.8) must obviously be understood as describing an idealized case, namely when defects altering and deforming ripple's phase are absent.

As a final comment, we consider the limit $\tau = \Delta P = c_0 = 0$ of equation (3.1). It is a particularly interesting case since it is concerned with the so-called Willmore surfaces, whose study has been one of the favourite subjects of the differential geometry in the 1980s (see [12] and references therein). It then follows from (3.4) that in the Willmore limit the Weierstrass potential p should be a solution of

$$p_{xx} + 8p^3 = 0.$$

Formulae (3.7) and (3.8) thus allow us to construct two simple examples of non-compact Willmore surfaces corresponding to the developable case $u = |\psi|^2 + |\varphi|^2 = \text{constant}$. These developable Willmore surfaces are described by formulae (3.7) and (3.8), p being given by

$$p(x) = \pm C_1 s\,\mathrm{d}\left[4C_1(x - C_2), \frac{1}{\sqrt{2}}\right] \tag{3.9}$$

where C_1, C_2 are real constants. It seems that open Willmore surfaces of this type have never been discussed in the literature before.

4. Statistical mechanics of Canham–Helfrich membranes

In this section, some aspects concerning the thermal fluctuations of the Canham–Helfrich model will be considered. We shall calculate correlators within the GWR framework and shall discuss their main features. We shall restrict only to fluctuations which do not change the topology of membranes. Under this assumption, the Gauss–Bonnet theorem tells us that the Gaussian curvature term does not enter in the matter [38]. After neglecting also the pressure

term, the density of the Canham–Helfrich free energy (1.1) can therefore be thought of as a power expansion up to second order in the scalar mean curvature H, say

$$F_{\mathrm{CH}} = F_{\mathrm{area}} + F_{\mathrm{spon}} + F_{\mathrm{Willmore}} = \int [\mathrm{d}\Sigma]\{\mu + \eta H + \beta H^2\}. \tag{4.1}$$

In terms of the GWR (2.1), (2.2) the area, the spontaneous curvature and the Willmore rigidity free energy terms read respectively

$$\begin{aligned} F_{\mathrm{area}} &= 4\mu \int [\mathrm{d}^2x](|\psi|^2 + |\varphi|^2)^2 \\ F_{\mathrm{spon}} &= 4\eta \int [\mathrm{d}^2x]p(|\psi|^2 + |\varphi|^2) \\ F_{\mathrm{Willmore}} &= 4\beta \int [\mathrm{d}^2x]p^2 \end{aligned} \tag{4.2}$$

($\mathrm{d}^2x = \mathrm{d}x\,\mathrm{d}y$). Within our framework Lagrangian terms need to be introduced according to

$$\int [\mathrm{d}^2x]\left[\lambda\left(\frac{1}{2}\partial_x\psi - \frac{\mathrm{i}}{2}\partial_y\psi - p\varphi\right) + \sigma\left(\frac{1}{2}\partial_x\varphi + \frac{\mathrm{i}}{2}\partial_y\varphi + p\psi\right) + \mathrm{c.c.}\right] \tag{4.3}$$

where λ and σ are complex-valued functions. Hence, our study of statistical mechanics of rigid Canham–Helfrich membranes by means of a Weierstrass-type formalism proceeds formally starting from the effective free energy

$$F_{\mathrm{CH}}^{\mathrm{GWR}} = F_{\mathrm{area}} + F_{\mathrm{spon}} + F_{\mathrm{Willmore}} + F_L \tag{4.4}$$

where F_{area}, F_{spon} , F_{Willmore} and F_L are given by (4.2), (4.3).

In order to study fluctuations of Canham–Helfrich membranes, we need the one-loop expression for the free energy (4.4). In the one-loop approximation the dominant contributions to the free energy are supposed to come from fields $p_1, \psi_1, \varphi_1, \lambda_1, \sigma_1$ *close* to solutions to the classical equation of motion $p_0, \psi_0, \varphi_0, \lambda_0, \sigma_0$. Under such a hypothesis, it is sufficient to consider the local free energy terms which are quadratic in the fluctuations $p_1, \psi_1, \varphi_1, \lambda_1, \sigma_1$. In doing so, one has to pay attention to the invariance of (4.4) under the set of transformations corresponding to the two-dimensional conformal group. These transformations consist of the complex coordinate transformation $z \to \xi(z)$, corresponding to two real one-dimensional coordinate transformations, and have in fact a Weyl local rescaling effect on the surface metric which is cancelled by the local coordinate scaling. In order to deal with the residue of gauge invariance of the free energy (4.4) while focusing on perturbative effects, we exploit the background field method performed in a normal gauge. That is, we shall consider fluctuations of Weierstrass fields orthogonal to the vector fields generating the infinitesimal diffeomorphisms (in general, this does not imply that we are dealing with membrane fluctuations which are normal to the shape). Since the gauge is purely local, the Faddeev–Popov determinant plays no role in the calculations and a naive measure can be used for fluctuations. The one-loop energy of the model (4.4) is therefore completely determined by the quadratic expansion of terms (4.2), (4.3) around a membrane background configuration, that is

$$\begin{aligned} F_{\mathrm{area}}^{(2)} = 8\mu \int [\mathrm{d}^2x]\big[&(|\psi_0|^2 + |\varphi_0|^2)\big(\psi_R^2 + \psi_I^2 + \varphi_R^2 + \varphi_I^2\big) \\ &+ 2(\psi_{R,0}\psi_R + \psi_{I,0}\psi_I + \varphi_{R,0}\varphi_R + \varphi_{R,0}\varphi_R)^2\big] \end{aligned} \tag{4.5}$$

$$F_{\mathrm{spon}}^{(2)} = 4\eta \int [\mathrm{d}^2x][p_0(|\psi|^2 + |\varphi|^2) + 2p(\psi_{R,0}\psi_R + \psi_{I,0}\psi_I + \varphi_{R,0}\varphi_R + \varphi_{I,0}\varphi_I)] \tag{4.6}$$

$$F^{(2)}_{\text{Willmore}} = 4\beta \int [\mathrm{d}^2x] p^2 \tag{4.7}$$

$$\begin{aligned} F^{(2)}_L = \int [\mathrm{d}^2x] \Bigg[& \lambda \left(\frac{1}{2}\partial_x \psi - \frac{\mathrm{i}}{2}\partial_y \psi - p_0 \varphi - p \varphi_0 \right) \\ & + \sigma \left(\frac{1}{2}\partial_x \varphi + \frac{\mathrm{i}}{2}\partial_y \varphi + p_0 \psi + p \psi_0 \right) + p\,(\psi \sigma_0 - \varphi \lambda_0) + \text{c.c.} \Bigg] \end{aligned} \tag{4.8}$$

where the following notation has been employed: (a) fields have subscripts R and I referring to their real and imaginary parts respectively, (b) subscripts for fluctuations have been omitted. In the momentum representation, the result is

$$F^{(2)} = \int [\mathrm{d}^2k] \tilde{F}^{(2)}(k) = \int [\mathrm{d}^2k] \big[\tilde{F}^{(2)}_{\text{area}}(k) + \tilde{F}^{(2)}_{\text{spon}}(k) + \tilde{F}^{(2)}_{\text{Willmore}}(k) + \tilde{F}^{(2)}_L(k) \big] \tag{4.9}$$

where

$$\begin{aligned} \tilde{F}^{(2)}_{\text{area}}(k) = {} & 8\mu \{ (|\psi_0|^2 + |\varphi_0|^2)[\bar{\psi}_R(k)\psi_R(k) + \bar{\psi}_I(k)\psi_I(k) + \bar{\varphi}_R(k)\varphi_R(k) + \bar{\varphi}_I(k)\varphi_I(k)] \\ & + 2[\psi_{R,0}\psi_R(k) + \psi_{I,0}\psi_I(k) + \varphi_{R,0}\varphi_R(k) + \varphi_{R,0}\varphi_R(k)][\psi_{R,0}\bar{\psi}_R(k) \\ & + \psi_{I,0}\bar{\psi}_I(k) + \varphi_{R,0}\bar{\varphi}_R(k) + \varphi_{R,0}\bar{\varphi}_R(k)] \} \end{aligned} \tag{4.10}$$

$$\begin{aligned} \tilde{F}^{(2)}_{\text{spon}}(k) = {} & 4\eta \{ [p_0(\bar{\psi}_R(k)\psi_R(k) + \bar{\psi}_I(k)\psi_I(k) + \bar{\varphi}_R(k)\varphi_R(k) + \bar{\varphi}_I(k)\varphi_I(k))] \\ & + [\bar{p}_{(k)}(\psi_{R,0}\psi_R(k) + \psi_{I,0}\psi_I(k) + \varphi_{R,0}\varphi_R(k) + \varphi_{I,0}\varphi_I(k)) + \text{c.c.}] \} \end{aligned} \tag{4.11}$$

$$\tilde{F}^{(2)}_{\text{Willmore}}(k) = 4\beta\, \bar{p}(k) p(k) \tag{4.12}$$

$$\begin{aligned} \tilde{F}^{(2)}_L(k) = 2\,\text{Re}\, \Bigg\{ & \bar{\lambda}_R(k) \left[\frac{k_x}{2\mathrm{i}}\psi_R(k) + \frac{k_y}{2\mathrm{i}}\psi_I(k) - p_0\varphi_R(k) - p(k)\varphi_{R,0} \right] \\ & - \bar{\lambda}_I(k) \left[-\frac{k_y}{2\mathrm{i}}\psi_R(k) + \frac{k_x}{2\mathrm{i}}\psi_I(k) - p_0\varphi_I(k) - p(k)\varphi_{I,0} \right] \\ & + \bar{\sigma}_R(k) \left[\frac{k_x}{2\mathrm{i}}\varphi_R(k) - \frac{k_y}{2\mathrm{i}}\varphi_I(k) + p_0\psi_R(k) + p(k)\psi_{R,0} \right] \\ & + \bar{\sigma}_I(k) \left[-\frac{k_y}{2\mathrm{i}}\varphi_R(k) - \frac{k_x}{2\mathrm{i}}\varphi_I(k) - p_0\psi_I(k) - p(k)\psi_{I,0} \right] \\ & + \bar{p}(k)[\sigma_{R,0}\psi_R(k) - \sigma_{I,0}\psi_I(k) - \lambda_{R,0}\varphi_R(k) + \lambda_{I,0}\varphi_I(k)] \Bigg\} \end{aligned} \tag{4.13}$$

($k = k_x + \mathrm{i}k_y$). Starting from (4.9)–(4.13) and carrying out the calculations, it turns out that all two-point correlators involving only the Weierstrass *geometric* fields ψ, φ, p, can be expressed according to the following simple scheme

$$\langle \bar{\phi}_i(k)\, \phi_j(k) \rangle = K_B T \frac{\bar{N}_i(k) N_j(k)}{D(k)} \tag{4.14}$$

where

$$\begin{aligned} & N_p(k) = \left(|k|^2 - 4p_0^2 \right) \\ & N_{\psi_R}(k) = 4p_0\, \text{Re}\, (\psi_0) + 2\mathrm{i}\, \text{Re}\, [(k_x + \mathrm{i}k_y)\varphi_0] \\ & N_{\psi_I}(k) = 4p_0\, \text{Im}\, (\psi_0) + 2\mathrm{i}\, \text{Im}\, [(k_x + \mathrm{i}k_y)\varphi_0] \\ & N_{\varphi_R}(k) = 4p_0\, \text{Re}\, (\varphi_0) - 2\mathrm{i}\, \text{Re}\, [(k_x + \mathrm{i}k_y)\bar{\psi}_0] \\ & N_{\varphi_I}(k) = 4p_0\, \text{Im}\, (\varphi_0) + 2\mathrm{i}\, \text{Im}\, [(k_x + \mathrm{i}k_y)\bar{\psi}_0] \end{aligned} \tag{4.15}$$

$$D(k) = 8\big\{\beta\big(|k|^2 - 4p_0^2\big)^2 + 2p_0\big(|k|^2 - 4p_0^2\big)\,\mathrm{Re}\,(\sigma_0\psi_0 - \lambda_0\varphi_0) + 8\mu(|\psi_0|^2 + |\varphi_0|^2)^2 \times \big(|k|^2 + 12p_0^2\big) + 4\eta p_0(|\psi_0|^2 + |\varphi_0|^2)\big(3|k|^2 - 4p_0^2\big)\big]\big\} \tag{4.16}$$

and $|k|^2 = k_x^2 + k_y^2$. We stress the generality of formulae (4.14)–(4.16): they give correlators in one-loop approximations following from the *whole* free energy (4.1) and for *all* possible backgrounds. All *geometric* correlators turn out to be therefore of the form $N_g/D(k)$ where the N_g are rather simple functions of slow-field components (i.e. of the membrane's background geometry) and momenta. In our approach, we have a straight geometrical interpretation of formulae since the p_0 field acts as a link between the extrinsic and intrinsic geometry of membranes (see equation (2.5)). The setting we adopted therefore enables us to discuss some properties of the Canham–Helfrich model with wide geometrical generality. This circumstance allows us, in particular, to argue on the validity of the one-loop approximation and on the infrared behaviour of the model formulating the discussion in terms of the couplings and of the geometric characteristics of the membrane background.

In formulae (4.14)–(4.16) for two-point functions, couplings β, η and μ enter by means of $D(k)$. As expected, the tension term can be neglected when describing the short wavelength fluctuations. However, its influence on the high wavelength fluctuations of the background geometry could also be rather weak if the spontaneous curvature η gives a positive contribution. In principle, the background field method would hold for weak couplings. As regards the rigid membrane model, it is also commonly meant that the one-loop approximation makes sense for momenta much larger than the curvatures (namely fluctuations small with respect to the radii of curvatures) of the background membrane geometry. We are now in a position to put these statements into a more precise form. This step is certainly of general interest when referring to concrete physical applications for which systems are expected to be well described by a rigid surface action. It is concerned with constraints expressing under which condition the fluctuations generated either by the intrinsic area term or by the extrinsic curvature term can be neglected with respect to the other. Tuning to zero some of the couplings enables one to simplify the matter, thus obtaining great advantages, as is discussed later. On general grounds, one obviously expects the suppression of the intrinsic area action terms to turn out to be natural for large momenta. In contrast, the infrared picture is not so clear. Note however that in the minimal background case geometric two-point correlators turn out to be of the type

$$\frac{N^*[k, \psi_0, \varphi_0]}{\beta|k|^2 + 8\mu(|\psi_0|^2 + |\varphi_0|^2)^2}. \tag{4.17}$$

They are thus free of infrared divergences and exhibit a massive pole at

$$8\beta^{-1}\mu(|\psi_0|^2 + |\varphi_0|)^2 = 2\beta^{-1}\mu\sqrt{g_0}.$$

More generally, it would be advisable to explicitly define some ranges of validity of the one-loop approximation in terms both of background geometry and of couplings β, η and μ. The characterization of the order of magnitude of the infrared regulator, hereinafter denoted as $\tilde{\Lambda}$, relies on the behaviour of $D(k)$. By considering the first variation of (4.4) with respect to p, one may employ the relation $\mathrm{Re}\,(\lambda_0\varphi_0 - \sigma_0\psi_0) = 4\beta p_0 + 2\eta u_0$ and recast $D(k)$ as

$$D(k) = 8\big\{\beta|k|^4 - 8\big(2\beta p_0^2 - \eta p_0 u_0 - \mu u_0^2\big)|k|^2 + 48p_0^2\big(\beta p_0^2 + 2\mu u_0^2\big)\big\} \tag{4.18}$$

where $u_0 = (|\psi_0|^2 + |\varphi_0|^2)$. Note that (4.18) can be put in a more geometrically transparent form in terms of the mean curvature H_0 and metric $\sqrt{g_0} = 4u_0^2$ of the background:

$$D(k) = 8\big\{\beta|k|^4 - 2\sqrt{g_0}\big(2\beta H_0^2 - \eta H_0 - \mu\big)|k|^2 + 3g_0 H_0^2\big(\beta H_0^2 + 2\mu\big)\big\}. \tag{4.19}$$

The roots of (4.18) are defined via

$$D(k) = 8\beta\big(|k| - r_1^-\big)\big(|k| - r_1^+\big)\big(|k| - r_2^-\big)\big(|k| - r_2^+\big) \tag{4.20}$$

where

$$r_1^{\pm} = \pm\sqrt{\frac{a-\sqrt{a^2-\beta b}}{\beta}} \qquad r_2^{\pm} = \pm\sqrt{\frac{a+\sqrt{a^2-\beta b}}{\beta}} \tag{4.21}$$

and

$$a = 4\big(2\beta p_0^2 - \eta p_0 u_0 - \mu u_0^2\big) = \sqrt{g_0}\big(2\beta H_0^2 - \eta H_0 - \mu\big)$$
$$b = 48 p_0^2\big(\beta p_0^2 + 2\mu u_0^2\big) = 3 g_0 H_0^2\big(\beta H_0^2 + 2\mu\big).$$

Hence, the infrared regulator $\tilde{\Lambda}$ is required only when the inequality $a \geqslant \sqrt{\beta b}$ (>0) holds. In this case one thus gets $\tilde{\Lambda} > r_2^+$. For non-minimal backgrounds, the simplest case to deal with corresponds to the negligible spontaneous curvature, $\eta \to 0$. It can be readily seen that in such a case (4.18) is strictly positive for every value of $|k|$ whenever $\beta H_0^2 < \mu(5+2\sqrt{6})$. It is also easily seen that this is just the case which arises when studying thermal fluctuations around a standard cylindrical background membrane satisfying the shape equation, for instance. In contrast, when $\beta H_0^2 \geqslant \mu(5+2\sqrt{6})$ then an infrared regulator $\tilde{\Lambda} > r_2^+$ needs to be introduced. This might be the case for a spherical membrane, for example. Indeed, in the case of vanishing spontaneous curvature the sphere of radius R is a solution to the shape equation when $R = H^{-1} = 2\mu/\Delta P$. Hence, for $(\Delta P)^2 > 4\beta^{-1}\mu^3(5+2\sqrt{6})$ the infrared regulator

$$\tilde{\Lambda} > r_2^+ = \frac{1}{2\mu}\sqrt{2(\Delta P)^2 - 4\mu^3\beta^{-1} + \sqrt{(\Delta P)^4 - 40\mu\beta^{-1}(\Delta P)^2 + 16\mu^6\beta^{-2}}}$$

must be introduced (we choose the $\sqrt{g_0} = 1$ local parametrization).

Another simple, but still interesting, case is when the energy area term can be neglected, that is $\mu \to 0$. In this case one has also to bear in mind that the spontaneous curvature η may exhibit a negative sign. So, it turns out that the reality of r_2^+ requires $H_0 \notin [0, (2+\sqrt{3})\eta\beta^{-1})$ for $\eta > 0$ and $H_0 \notin ((2+\sqrt{3})\eta\beta^{-1}, 0]$ for $\eta < 0$. In these cases the constraint $\tilde{\Lambda} > r_2^+$ must therefore be taken into account.

Once the range of validity of the model has been clarified, it makes sense to consider effects of fluctuations on the membrane geometry. For instance, we can consider the quantity $\frac{\delta\sqrt{g}}{\sqrt{g_0}}$, which gives a measure of thermal fluctuations of the membrane metric as compared to the metric of the membrane background. One has

$$\begin{aligned}
\frac{\delta\sqrt{g}}{\sqrt{g_0}} = \frac{\delta\sqrt{u^2}}{\sqrt{u_0^2}} &= 8K_BT\int[\mathrm{d}^2k]\frac{\big(|k|^2+12p_0^2\big)}{D(k)} \\
&= \frac{K_BT}{8\pi\beta}\left\{\ln(\beta|k|^4 - 2a|k|^2 + b) - \frac{\big(2a+24\beta p_0^2\big)}{\sqrt{a^2-\beta b}}\tanh^{-1}\left[\frac{\beta|k|^2-a}{\sqrt{a^2-\beta b}}\right]\right\}_{\tilde{\Lambda}}^{|k|_{\max}} \\
&= \frac{K_BT}{8\beta\pi}\left\{\ln\big[\beta|k|^4 - 2\sqrt{g_0}\big(2\beta H_0^2 - \eta H_0 - \mu\big)|k|^2 + 3g_0H_0^2\big(\beta H_0^2+2\mu\big)\big]\right. \\
&\quad + \frac{2\big(\mu+\eta H_0 - 5\beta H_0^2\big)}{\sqrt{\beta^2H_0^4 - 4\beta\eta H_0^3 + (\eta^2-10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}} \\
&\quad \left.\times\tanh^{-1}\left[\frac{\beta|k|^2 + \sqrt{g_0}\big(\mu+\eta H_0 - 2\beta H_0^2\big)}{\sqrt{g_0}\sqrt{\beta^2H_0^4 - 4\beta\eta H_0^3 + (\eta^2-10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}}\right]\right\}_{\tilde{\Lambda}}^{|k|_{\max}}.
\end{aligned} \tag{4.22}$$

By making use of the formula (4.22), the effective surface tension generated by thermal fluctuation can be introduced. The renormalization of the membrane tension can be derived according to

$$\mu_{\rm eff} = \mu\left(1 + \frac{\delta\sqrt{g}}{\sqrt{g_0}}\right). \tag{4.23}$$

While pointing out once again the generality of the approach, we also note that taking only the ultraviolet dominant term we would get

$$\mu_{\rm eff} = \mu\left[1 + \frac{K_B T}{2\pi\beta}\ln\frac{|k|_{\max}}{\tilde{\Lambda}}\right]. \tag{4.24}$$

This is exactly the result for the effective tension which is obtained after evaluating in the Monge representation the contribution from the area fluctuations induced just by the energy term $\beta\int[{\rm d}\Sigma]H^2$, see, e.g., [16] (notational differences w.r.t. formula (1) in [16] are reconciled by means of the substitutions $|k|_{\max} = a^{-1}$, $\tilde{\Lambda} = L^{-1}$ and $\kappa = \beta/2$).

Among the physical consequences of the thermal fluctuation there is an increase in the effective local bending energy. Precisely, one gets

$$\begin{aligned}
\delta(H^2\sqrt{g}) &= \int[{\rm d}^2k]\langle\bar{p}(k)p(k)\rangle = \frac{K_B T}{2\pi}\int{\rm d}|k|\frac{|k|\left(|k|^2 - 4p_0^2\right)^2}{D(k)} \\
&= \frac{4K_B T}{\pi}\left\{\frac{|k|^2}{2\beta} + \frac{\left(a - 4\beta p_0^2\right)}{2\beta^2}\ln D(k)\right. \\
&\quad \left. - \frac{a^2 - b\beta + \left(a - 4\beta p_0^2\right)^2}{2\beta^2\sqrt{a^2 - b\beta}}\tanh^{-1}\left[\frac{\beta|k|^2 - a}{\sqrt{a^2 - \beta b}}\right]\right\}_{\tilde{\Lambda}}^{|k|_{\max}} \\
&= \frac{4K_B T}{\pi}\left\{\frac{|k|^2}{2\beta} + \frac{\sqrt{g_0}\left(\beta H_0^2 - \eta H_0 - \mu\right)}{2\beta^2}\right. \\
&\quad \times\ln\left\{8\left[\beta|k|^4 - 2\sqrt{g_0}\left(2\beta H_0^2 - \eta H_0 - \mu\right)|k|^2 + 3g_0 H_0^2\left(\beta H_0^2 + 2\mu\right)\right]\right\} \\
&\quad - \frac{\sqrt{g_0}\left[\beta^2 H_0^4 - 3\beta\eta H_0^3 + (\eta^2 - 6\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2\right]}{\beta^2\sqrt{\beta^2 H_0^4 - 4\beta\eta H_0^3 + (\eta^2 - 10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}} \\
&\quad \left.\times\tanh^{-1}\left[\frac{\beta|k|^2 + \sqrt{g_0}\left(\mu + \eta H_0 - 2\beta H_0^2\right)}{\sqrt{g_0}\sqrt{\beta^2 H_0^4 - 4\beta\eta H_0^3 + (\eta^2 - 10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}}\right]\right\}_{\tilde{\Lambda}}^{|k|_{\max}}.
\end{aligned}$$

The quantity $\delta(H^2\sqrt{g})$ thus behaves as the sum of a leading term $|k|^2$ plus logarithmic correction. So we obtain, for instance, that for minimal backgrounds ($H_0 = 0$) the increase in the curvature energy due to fluctuations is given by

$$4\beta\int[{\rm d}^2k]\langle\bar{p}(k)p(k)\rangle = \frac{8K_B T}{\pi\beta}\left\{\beta|k|^2 - 2\mu\sqrt{g_0}\ln(\beta|k|^2 + 2\mu\sqrt{g_0})\right\}_{\tilde{\Lambda}}^{|k|_{\max}}.$$

(Recall that in this case we can even set $\tilde{\Lambda} = 0$.)

Finally, let us consider the problem of renormalizing the spontaneous curvature. Calculation of $\delta(H\sqrt{g})$ yields

$$\begin{aligned}
\delta(H\sqrt{g}) = 4\delta(pu) &= \frac{8K_B T p_0 u_0}{\pi}\int{\rm d}|k|\frac{|k|\left(3|k|^2 - 4p_0^2\right)}{D(k)} \\
&= \frac{K_B T p_0 u_0}{\pi}\left\{3\ln D(k) + \frac{8\beta p_0^2 - 6a}{\sqrt{a^2 - b\beta}}\tanh^{-1}\left[\frac{\beta|k|^2 - a}{\sqrt{a^2 - \beta b}}\right]\right\}_{\tilde{\Lambda}}^{|k|_{\max}}
\end{aligned}$$

that is

$$\delta(H\sqrt{g}) = \frac{K_B T H_0 \sqrt{g_0}}{4\pi} \left\{ 3\ln\left[\beta|k|^4 - 2\sqrt{g_0}\left(2\beta H_0^2 - \eta H_0 - \mu\right)|k|^2 + 3g_0 H_0^2\left(\beta H_0^2 + 2\mu\right)\right] - 2\frac{\left(5\beta H_0^2 - 3\eta H_0 - 3\mu\right)}{\sqrt{\beta^2 H_0^4 - 4\beta\eta H_0^3 + (\eta^2 - 10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}} \times \tanh^{-1}\left[\frac{\beta|k|^2 + \sqrt{g_0}\left(\mu + \eta H_0 - 2\beta H_0^2\right)}{\sqrt{g_0}\sqrt{\beta^2 H_0^4 - 4\beta\eta H_0^3 + (\eta^2 - 10\mu\beta)H_0^2 + 2\eta\mu H_0 + \mu^2}}\right] \right\}_{\tilde{\Lambda}}^{|k|_{\max}}. \tag{4.25}$$

Since $\delta(H\sqrt{g}) \propto H_0\sqrt{g_0}$ then one-loop thermal fluctuations of minimal backgrounds do not contribute to the renormalization of spontaneous curvature. That is, in such a case η_{eff} is actually just η. A similar result has been obtained in [39] under the hypothesis of weak background bends (which corresponds to an almost minimality condition). For arbitrary non-minimal backgrounds the formula (4.25) can be used to evaluate the renormalized spontaneous curvature according to

$$\eta_{\text{eff}} = \eta\left[1 + \frac{\delta(H\sqrt{g})}{H_0\sqrt{g_0}}\right]. \tag{4.26}$$

This formula shows us how thermal fluctuations affect the spontaneous curvature of membranes with strong background bends.

Analysing the Lagrangian sector of correlators is a more difficult task. The non-approximate form of two-point functions involving the Lagrange fields λ and σ is, unfortunately, really hard to deal with. All of them are of the form $N_L(k)/D_L(k)$ where the N_L are some complicated functions of the slow fields and k while

$$D_L(k) = \left(|k|^2 - 4p_0^2\right)^2 D(k). \tag{4.27}$$

For instance, it results in

$$\langle\bar{p}(k)\,\lambda_R(k)\rangle = \left\{2\left(|k|^2 - 4p_0^2\right)[2p_0\,\mathrm{Re}\,\lambda_0 + \mathrm{i}\,\mathrm{Re}\,(k\sigma_0)] + 32\mu u_0\left[8\mathrm{i}p_0\,\mathrm{Re}\,(k\bar{\psi}_0) - \left(|k|^2 + 12p_0^2\right)\mathrm{Re}\,\varphi_0\right] + 8\eta\left[\mathrm{i}\left(|k|^2 + 4p_0^2\right)\mathrm{Re}\,(k\bar{\psi}_0) - 4p_0|k|^2\,\mathrm{Re}\,\varphi_0\right]\right\}/D(k) \tag{4.28}$$

$$\langle\bar{p}(k)\lambda_I(k)\rangle = \left\{2\left(|k|^2 - 4p_0^2\right)[2p_0\,\mathrm{Im}\,\lambda_0 + \mathrm{i}\,\mathrm{Im}\,(k\sigma_0)] + 32\mu u_0\left[8\mathrm{i}p_0\,\mathrm{Im}\,(k\bar{\psi}_0) + \left(|k|^2 + 12p_0^2\right)\mathrm{Im}\,\varphi_0\right] + 8\eta\left[\mathrm{i}\left(|k|^2 + 4p_0^2\right)\mathrm{Im}\,(k\bar{\psi}_0) + 4p_0|k|^2\,\mathrm{Im}\,\varphi_0\right]\right\}/D(k) \tag{4.29}$$

$$\langle\bar{p}(k)\sigma_R(k)\rangle = \left\{2\left(|k|^2 - 4p_0^2\right)[2\,\mathrm{Re}\,\sigma_0 - \mathrm{i}\,\mathrm{Re}\,(k\lambda_0)] + 32\mu u_0\left[8\mathrm{i}p_0\,\mathrm{Re}\,(k\varphi_0) + \left(|k|^2 + 12p_0^2\right)\mathrm{Re}\,\psi_0\right] + 8\eta\left[\mathrm{i}\left(|k|^2 + 4p_0^2\right)\mathrm{Re}\,(k\varphi_0) + 4p_0|k|^2\,\mathrm{Re}\,\psi_0\right]\right\}/D(k) \tag{4.30}$$

$$\langle\bar{p}(k)\sigma_I(k)\rangle = \left\{2\left(|k|^2 - 4p_0^2\right)[2\,\mathrm{Re}\,\sigma_0 - \mathrm{i}\,\mathrm{Re}\,(k\lambda_0)] + 32\mu u_0\left[8\mathrm{i}p_0\,\mathrm{Im}\,(k\varphi_0) + \left(|k|^2 + 12p_0^2\right)\mathrm{Im}\,\psi_0\right] + 8\eta\left[\mathrm{i}\left(|k|^2 + 4p_0^2\right)\mathrm{Im}\,(k\varphi_0) + 4p_0|k|^2\,\mathrm{Im}\,\psi_0\right]\right\}/D(k) \tag{4.31}$$

and

$$\langle\bar{\psi}_R(k)\lambda_R(k)\rangle = \left\{\beta N^{\beta}_{\bar{\psi}_R\lambda_R} + 32\mu u_0 N^{\mu}_{\bar{\psi}_R\lambda_R} + 4\eta N^{\eta}_{\bar{\psi}_R\lambda_R} + N^{0}_{\bar{\psi}_R\lambda_R}\right\}\Big/ D^*(k) \tag{4.32}$$

where

$$N^{\beta}_{\bar{\psi}_R\lambda_R} = -8\mathrm{i}\left(|k|^2 - 4p_0^2\right)^2 \operatorname{Re} k$$

$$\begin{aligned} N^{\mu}_{\bar{\psi}_R\lambda_R} = {} & 64u_0\{\mathrm{i}|k|^2[u_0 \operatorname{Re} k - \operatorname{Re}\varphi_0 \operatorname{Re}(k\varphi_0)] - 24p_0^3 \operatorname{Re}\psi_0 \operatorname{Re}\varphi_0 \\ & + 4\mathrm{i}p_0^2[4 \operatorname{Re}\psi_0 \operatorname{Re}(k\bar{\psi}_0) + 3 \operatorname{Re}\varphi_0 \operatorname{Re}(k\varphi_0) - 3u_0 \operatorname{Re} k] \\ & + 2p_0[4 \operatorname{Re}(k\bar{\psi}_0) \operatorname{Re}(k\varphi_0) - |k|^2 \operatorname{Re}\psi_0 \operatorname{Re}\varphi_0]\} \end{aligned}$$

$$\begin{aligned} N^{\eta}_{\bar{\psi}_R\lambda_R} = {} & 16|k|^2 \operatorname{Re}(k\bar{\psi}_0) \operatorname{Re}(k\varphi_0) + 128\mathrm{i}p_0^3[u_0 \operatorname{Re} k + \operatorname{Re}\psi_0 \operatorname{Re}(k\bar{\psi}_0)] \\ & + 64p_0^2[\operatorname{Re}(k\bar{\psi}_0) \operatorname{Re}(k\varphi_0) - 2|k|^2 \operatorname{Re}\varphi_0 \operatorname{Re}\psi_0] \\ & + 32\mathrm{i}p_0|k|^2[\operatorname{Re}\psi_0 \operatorname{Re}(k\bar{\psi}_0) + 2 \operatorname{Re}\varphi_0 \operatorname{Re}(k\varphi_0) - 3u_0 \operatorname{Re} k] \end{aligned}$$

$$\begin{aligned} N^{0}_{\bar{\psi}_R\lambda_R} = {} & 4\left(|k|^2 - 4p_0^2\right)\{[2p_0 \operatorname{Re}\lambda_0 - \mathrm{i} \operatorname{Re}(k\sigma_0)][2p_0 \operatorname{Re}\psi_0 + \mathrm{i} \operatorname{Re}(k\varphi_0)] \\ & + 2 \operatorname{Re} k \operatorname{Re}(\lambda_0\varphi_0 - \sigma_0\psi_0)\} \end{aligned}$$

and $D^*(k) = \left(|k|^2 - 4p_0^2\right)D(k)$.

Purely Lagrangian correlators are more complicated and we do not report them. The important feature to point out is that the positivity of the denominator (4.27) may not be enough to ensure the reality in configuration space of Lagrangian two-point functions, not even for rather high momenta. A little inspection reveals that the effect is induced mainly by the tension and the spontaneous curvature. If we skip this aspect with the reasonable motivation that Lagrangian multiplier fields are not strictly related to physical observables and should not have necessarily observable excitations, then the infrared regulator of the model is required to satisfy the conditions previously discussed for the geometric Weierstrass fields. However, this approach may be somewhat elusive and unsatisfactory in that fluctuations of Lagrangian fields are commonly considered short range. In order to make a more concrete analysis, it is useful to simplify the matter by restricting ourselves to minimal backgrounds, $p_0 = 0$, and considering some typical propagators. Within such a minimal background limit, the result is

$$\begin{aligned} \langle\bar{\lambda}_R(k)\lambda_R(k)\rangle_{p_0=0} = {} & \left\{|k|^2\left[(\operatorname{Re} k\sigma_0)^2 + 256\mu^2u_0^2(\operatorname{Re}\varphi_0)^2\right]\right. \\ & - 32\mu\left(\beta|k|^2 + 8\mu u_0^2\right)[2(\operatorname{Re} k\bar{\psi}_0)^2 + u_0|k|^2)] \\ & \left. + 8\eta|k|^2 \operatorname{Re}(k\bar{\psi}_0)[2\eta \operatorname{Re}(k\bar{\psi}_0) + \operatorname{Re}(k\sigma_0)])\right\}\Big/ D^0 \end{aligned}$$

$$\begin{aligned} \langle\bar{\lambda}_I(k)\lambda_I(k)\rangle_{p_0=0} = {} & \left\{|k|^2\left[(\operatorname{Im} k\sigma_0)^2 + 16\mu^2u_0^2(\operatorname{Im}\varphi_0)^2\right]\right. \\ & - 2\mu\left(\beta|k|^2 + 8\mu u_0^2\right)[2(\operatorname{Im} k\bar{\psi}_0)^2 + u_0|k|^2] \\ & \left. - 8\eta|k|^2 \operatorname{Im}(k\bar{\psi}_0)[2\eta \operatorname{Im}(k\bar{\psi}_0) - \operatorname{Im}(k\sigma_0)])\right\}\Big/ D^0 \end{aligned}$$

$$\begin{aligned} \langle\bar{\sigma}_R(k)\sigma_R(k)\rangle_{p_0=0} = {} & \left\{|k|^2\left[(\operatorname{Re} k\bar{\lambda}_0)^2 + 16\mu^2u_0^2(\operatorname{Re}\psi_0)^2\right]\right. \\ & - 2\mu\left(\beta|k|^2 + 8\mu u_0^2\right)[2(\operatorname{Re} k\varphi_0)^2 + u_0|k|^2] \\ & \left. + 8\eta|k|^2 \operatorname{Re}(k\varphi_0)[2\eta \operatorname{Re}(k\varphi_0) - \operatorname{Re}(k\bar{\lambda}_0)]\right\}\Big/ D^0 \end{aligned}$$

$$\langle \bar{\sigma}_I(k)\sigma_I(k)\rangle_{p_0=0} = \big\{|k|^2\big[(\operatorname{Im} k\bar{\lambda}_0)^2 + 16\mu^2 u_0^2(\operatorname{Im}\psi_0)^2\big] - 2\mu\big(\beta|k|^2 + 8\mu u_0^2\big)[2(\operatorname{Im} k\varphi_0)^2 + u_0|k|^2)] - 8\eta|k|^2\operatorname{Im}(k\varphi_0)[2\eta\operatorname{Im}(k\varphi_0) + \operatorname{Im}(k\bar{\lambda}_0)]\big\}\big/ D^0$$

with $D^0 = 2|k|^4\big[\beta|k|^2 + 8\mu u_0^2\big]$. In the ultraviolet limit, one gets in particular

$$\langle \bar{\lambda}_R(k)\,\lambda_R(k)\rangle_{p_0=0} = \frac{1}{2\beta|k|^4}\{[(\operatorname{Re} k\sigma_0)^2] - 32\mu\beta[2(\operatorname{Re} k\bar{\psi}_0)^2 + u_0|k|^2] + 16\eta\operatorname{Re}(k\bar{\psi}_0)[\eta\operatorname{Re}(k\bar{\psi}_0) + \operatorname{Re}(k\sigma_0)]\} \tag{4.33}$$

etc. Note that these expressions also follow avoiding terms of the second order with respect to the tension. Equation (4.33) shows us that the one-loop approximation for the full action (4.4) holds only in a small coupling limit provided that (4.33), as well as all the similar formulae for the other Lagrangian fields, is positive.

5. Conclusions

The use of geometrical methods in theoretical biophysics and physics of liquid interfaces is well established [1–8]. Several experiments on membranes find a natural interpretation in terms of simplified geometrical models. Models describing these systems as two-dimensional surfaces fluctuating in a bulk space have proved to be extremely successful and their various problems and features have therefore been studied and analysed [1–8].

In this paper, we have attempted to elucidate some basic features related both to the geometrical modelling and to the statistical behaviour of rigid membranes described by a free energy of the Canham–Helfrich type. To start with, formulae for some lamellar periodical configurations for Canham–Helfrich rigid membranes have been given. Next, we have tried to obtain a more systematic understanding of perturbative aspects of the model. Concerning thermal effects, all results available in the literature actually refer to momenta in ultraviolet ranges, that is for fluctuations of very short wavelength. In order to get an insight into the infrared region, here we have considered the problem of investigating the full action (4.1) with a statistical measure properly constrained by the equations relating the extrinsic and the intrinsic geometrical quantities of the membrane surface. Having been guided by the consideration that in order to attack this fascinating topic all techniques and ideas should be carefully considered, we worked in the framework of the generalized Weierstrass representation for surfaces in $\mathbb{R}^3$ introduced in [18]. Basically, the route is very close to the idea advocated in [16] to use a mean curvature measure (H-measure). However, the measure induced by the GWR, which we can even call the $H\sqrt[4]{g}$-measure (see equations (2.5) and (4.2)), has the net advantage of providing a natural way to take into account simultaneously both the metric and the mean curvature fluctuations. From a technical point of view, the main features concerning the description of rigid membranes through this representation are that compatibility conditions defining the immersions are of linear type and that the description is intrinsic. Although, in principle, the same calculations could have been carried out by making use of other approaches (e.g., the Monge representation), the adoption of the GWR seems to be the most advantageous to get results with wide generality since it is based on simple formulae for geometrical quantities defining the local membrane energy (the square root for the area term and high-order derivative for the mean curvature terms are in fact avoided). Furthermore, the compatibility conditions read very simply as well, being nothing but the couple of equations (2.1). By virtue of these features, the GWR exhibits a computational efficiency for the study of thermal fluctuations since it introduces a very simple scheme in Fourier space for two-point functions of Weierstrass

fields, equations (4.14)–(4.16), (4.18). We can therefore exploit a very powerful formulation which, being in fact intrinsic (configuration space coordinates X^μ do not enter the matter), straightforwardly allows us to make the role of the background membrane geometry very explicit, even in cases of non-trivial topology (which in our study is still demanded not to change under thermal fluctuation). As argued before, this circumstance allows us to express relevant formulae, and to discuss their main features, directly in terms of the mean curvature and of the local metric of background geometry. A new light has thus been shed on the characterization of the infrared regulator and on the renormalization of couplings.

Certainly many problems still remain in obtaining a more complete understanding of the membrane systems, since one is necessarily confronted with the complexity of these systems which may limit the geometrical description program as a very tentative one. We hope, however, that the procedure we outlined above may be useful in giving some further insight into the subject.

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Appendix. Comments on constant mean curvature membrane geometries

The aim of this appendix is to briefly recall some results which are known from the differential geometry and concern one of the most relevant classes of surfaces, that is the *constant mean curvature surfaces* (see, e.g., [13]). These surfaces provide a natural class of solutions to the shape equation. Among others, they look suitable for membrane multilayer systems since it is a classical result that a surface parallel to a constant mean curvature (CMC) surface and lying at distance $1/H_0$ is itself a CMC surface. Although CMCs have been widely discussed in the context of membrane physics, they have been mainly taken into account in a rather old-fashioned way, whereas the last two decades have seen much in this subject. It is worth pointing out the existence of results which, even being potentially interesting and representative, received quite poor attention from the point of view of applications to membrane models. Undoubtedly, the main breakthrough in the modern differential geometry of CMCs has been the discovery of the so-called Wente tori [40] which provide us a counterexample to the long-standing Hopf conjecture that the sphere was the only example of boundaryless compact constant mean curvature surface. Remarkably, Bobenko succeeded in giving a description of all CMC tori in terms of θ-functions [41]. These results, jointly with the development of *gluing construction* methods, which allow one to construct CMC surfaces starting from a collection of previously known ones, and the *conjugate surface* method, which allows one to construct CMC surfaces by finding appropriate conjugate minimal surfaces on the 3-sphere $\mathbb{S}^3$, have led to the construction and characterization of several CMC surfaces [42–46]. The use of all these results, especially from the point of view of the application of numerical methods to analyse the stability of membrane configurations, would definitily be of great interest. Such an activity is apparently missing.

In terms of the GWR, arbitrary surfaces of constant nonvanishing mean curvature H_0 can be generated by the integral formulae (2.2) where Weierstrass fields ψ and φ now satisfy

the differential system (2.1) with $p = (|\psi|^2 + |\varphi|^2)H_0$. This system has been discussed in wide generality in [23] (see also [21, 24–28]). It is equivalent to the completely integrable Euclidean σ-model in two dimensions, the topological charge of the model being just the CMC topological invariant integral curvature [27]. In order to model CMC Canham–Helfrich membranes one has clearly to consider also the shape equation (3.1), which now enters as the constraint

$$\kappa_c(2H_0 + c_0)[(2H_0 - c_0)H_0 - 2K] - 2\tau H_0 + \Delta P = 0. \tag{5.1}$$

Depending on the value of spontaneous curvature, two main cases can thus be distinguished, namely $H_0 = -c_0/2$ and $H_0 \neq -c_0/2$. When $H_0 = -c_0/2$, every CMC surface is a solution to the shape equation (3.1) provided that $\Delta P = 2\tau H_0$. In this case a number of geometric situations are therefore involved (see also the CMC surfaces section at the GANG web archives [47]). The case $H_0 \neq -c_0/2$ requires more attention to be paid since the Gaussian curvature K is required to be constant too. In other words, both principal curvatures should be constant. This case is related to Weierstrass potentials of the type

$$p^2 = \frac{|\partial A(z)|^2}{(|A|^2 + \xi/2)^2}$$

where $A(z)$ is an arbitrary analytic function and $\xi = 2K_0H_0^{-2} = H_0^{-2}\big[\kappa_c^{-1}(\Delta P - 2\tau H_0)(2H_0 + c_0)^{-1} + (2H_0 - c_0)\big]$ [17]. In the case of CMC surfaces of vanishing Gaussian curvature, potentials are of the type $p \propto A(z)\,\overline{A(z)}$. The case of separable Weierstrass potential, $p \propto A(z)\overline{A(z)}$, is inclusive of surfaces of constant squared mean curvature density, $H^2\sqrt{g} = \text{const}$. They correspond to the simplest nonvanishing choice for the potential of the representation, i.e. $p = p_0 = \text{constant}$. The simplest element of this class is the standard cylinder of radius $R = (2H_0)^{-1}$, a case which has been largely discussed in the context of the Canham–Helfrich systems (see [8] and references therein). An extension of the standard cylinders to non-circular ones easily arises for Weierstrass fields of the trigonometric type

$$\psi = a\bar{A}\cos\left[2\,\mathrm{Re}\left(\int A^2\,\mathrm{d}z\right)\right] \qquad \varphi = -aA\sin\left[2\,\mathrm{Re}\left(\int A^2\,\mathrm{d}z\right)\right] \tag{5.2}$$

a and A being a real and a complex constant respectively. This case corresponds to a linear superposition of two Weierstrass solutions of the single wave-type (which is just the form for the standard cylinder) and provides us with CMC developable tubular surfaces whose sections are specified by the physical parameters via $\big(X_2 - X_2^0\big)^2 = \big(X_3 - X_3^0\big)\big[H_0^{-1} - \big(X_3 - X_3^0\big)\big]$.

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